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THE CORE MATCHINGS OF MARKETS WITH TRANSFERS

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Abstract

We characterize the structure of the set of core matchings of an assignment game (a two-sided market with transfers). Such a set satisfies a property we call *consistency*. Consistency of a set of matchings states that, for any matching ν, if, for each agent i there exists a matching μ in the set for which $\mu(i) = \nu(i)$, then ν is in the set. A set of matchings satisfies consistency if and only if there is an assignment game for which all elements of the set maximize the surplus. We also identify conditions under which we can assume the assignment game has nonnegative values.

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1 Introduction

In matching markets, the data we observe are the matchings themselves. Preferences of agents over each other, cardinal utility, or monetary transfers are typically not observed. In many matching markets, such as the market which matches hospitals to interns, no transfers in fact take place. But in other markets, such as housing markets, buyers and sellers are matched, and transfers do in fact take place. In yet other markets, such as organ donations, transfers may, but should not, take place. In this paper, we ask how we can empirically distinguish between these two types of markets with data on matchings alone.

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We study the testable implications of the standard model of two-sided markets with flexible prices: the so-called *assignment game*. The assignment game was introduced by Koopmans and Beckmann (1957) and Shapley and Shubik (1971). It is the basis for a body of modern economic theories; auction theory being the best known of these. Theoretical work on the assignment game has focused on the model's predicted utilities. Empirical work, on the other hand, deals almost exclusively with matchings, as utilities and transfers are often unobservable.

We characterize the sets of matchings (i.e. the data on who buys from whom) which can be generated by the model. An assignment game specifies two sets of economic agents, usually understood as buyers and sellers; but, following tradition, we will refer to men and women. Agents have quasilinear preferences over each other and money. Men can match to women; agents can also remain unmatched. Because of quasilinearity, each pair consisting of a man and a woman (a couple) generates some surplus. The games always have nonempty core. Core payoff vectors divide the maximal possible surplus among the set of agents. We will say a matching is a *core matching* if it is one which maximizes this surplus. Our aim in this paper is to understand the structure of the set of core matchings.

We propose a joint test on observed matchings of the hypotheses that i) utility is freely transferable and ii) matchings are chosen to maximize aggregate surplus. We do this by characterizing the exact structure of sets of core matchings of assignment games. If we therefore know the set of possible matchings that might arise, we can verify whether or not they could have been generated by a transferable utility model and surplus maximization.

We show that a set of matchings can be the set of core matchings for an assignment game if and only if a simple property, which we call *consistency*, is satisfied. Consistency of a set of matchings E states the following. Take any matching ν of men to women. Suppose that if a man is matched to a woman under ν, then there exists $\mu \in E$ which matches this man to the same woman. Suppose that if a man is unmatched under ν, there exists $\mu \in E$ for which he is unmatched. Suppose similar statements hold for women. If ν satisfies these properties, then ν must itself be an element of E. Consistency thus might be viewed as the following: allow each agent x to choose some $\mu_x \in E$. If the function $\nu(x) = \mu_x(x)$ is itself a matching, then ν must be an element of E as well.

Consistency is a necessary and sufficient condition for a set of matchings to be the core of some assignment game. In fact, more is true. Consistency is satisfied if and only if a set of matchings is the core of some assignment game with integer values. Obviously, if matchings are the only observable, these are the only things we can test. Our results illustrate that, from the point of view of observing matchings, the complete testable implications of the assignment game come in the form of consistency of the set of possible matchings.

Consistency has the property that for any set of matchings, there is a unique smallest consistent extension (with respect to set inclusion). The intersection of an arbitrary collection of sets of consistent matchings is itself consistent. This property is useful in environments in which we may observe a set of matchings E and want to verify whether they can be core matchings for some game. If we know that some matchings F are necessarily not elements of the core, then for E to be possible core matchings, we simply need to find the smallest consistent extension of E and verify that it is disjoint from F. For example, if we generate matchings which are patently inefficient (e.g. by breaking up matches which should be profitable), we can see if these can be generated from the matchings in E.

We also characterize the set of matchings which coincide with the core of an assignment game where all surpluses are nonnegative. This requires a somewhat more restrictive notion of consistency (which we term monotone consistency) which is nonetheless simple to verify.

Three aspects of our results are worth emphasizing.

Firstly, we provide the first characterization of core matchings in the assignment game. For the model without transferable utility, the Gale-Shapley marriage market, a characterization has been known for a long time (Knuth, 1976): the core matchings have a lattice structure. The lattice characterization has been very useful in the study of these markets. Our result is, in a sense, a counterpart to the lattice result for the model with transfers.

Secondly, a well-known mathematical observation is that if surpluses are drawn from an absolutely continuous distribution, then the set of core matchings is generically a singleton. To this end, many researchers have focused on the case in which in fact there is a unique core matching. Of course, there is no foundation for the hypothesis surpluses are drawn from an absolutely continuous distribution. We believe that the hypothesis is not justified empirically, or mathematically. In fact, casual observation implies that surpluses in fact come in discrete units (pennies, for example).

Thirdly, in order to practically use our results, we must be able to observe more than one matching. In practice, this may be difficult to do. In Section 5 we explain how to use our result in realistic econometric settings.

1.1 Related Literature

Shapley and Shubik (1971) first studied the core of assignment games. They establish results on the set of core imputations (utilities) in the associated transferable utility game. In particular, they characterize core imputations through a linear programming argument. This characterization implies that a matching is a core matching if and only if it supports any core imputation, with each couple sharing their surplus only amongst themselves. They also show that the set of core imputations restricted to M (or W)

form a lattice under the pointwise ordering. Shapley and Shubik present no results on the structure of core matchings.

More recently, Sotomayor (2003) (and later Wako (2006)) establish a relationship on the cardinality of the set of core matchings and the structure of the set of core imputations. In particular, these authors establish that if there is only one core matching, then the set of possible imputations is infinite (the converse is not true in general). Nuñez and Rafels (2006) study the dimension of the core in the space of imputations.

Related to this work is an earlier paper by Echenique (2008), which studies a matching model in which transfers cannot be made, but each side of the market has strict preferences over the other side. Echenique establishes conditions that are necessary and sufficient for a collection of matchings to be a *subset* of core matchings for some such preference profile. Strictness of preference in this environment is critical; if preferences are allowed to be weak, all sets of matchings can be the subset of a set of core matchings for some preference profile (the profile in which all agents are indifferent between everything). In our work, there is no trivial analogue of the statement that preferences are strict, and hence statements about subsets of core matchings require instead knowledge that some matchings cannot be core matchings.

Section 2 provides the model and main results, while Section 3 is devoted to proofs. Section 4 concludes.

2 The Model

Let M and W denote disjoint finite sets of agents. A ***matching*** is a function $\mu : M \cup W \to M \cup W$ such that for all $m \in M$, $\mu(m) \in W \cup \{m\}$, for all $w \in W$, $\mu(w) \in M \cup \{w\}$, and for all $i \in M \cup W$, $\mu(\mu(i)) = i$. An agent $i \in M \cup W$ is ***single in*** μ if $\mu(i) = i$.

If a matching μ satisfies the property that for all $m \in M$, $\mu(m) \in W$ and for all $w \in W$, $\mu(w) \in M$, we will say it is a ***complete matching***; *i.e.* a matching μ is complete if no agent is single in μ. Denote the set of matchings by $\mathcal{M}$ and the set of complete matchings by $\mathcal{M}_c$.

An ***assignment game*** α is a matrix $[\alpha_{m,w}] \in \mathbb{R}^{M \times W}$. The interpretation is that $\alpha_{m,w}$ is the surplus generated by m and w if they match. We will say an assignment game α is integer valued if for all $(m,w) \in M \times W$, $\alpha_{m,w} \in \mathbb{Z}$. We say it is nonnegative if for all $(m,w) \in M \times W$, $\alpha_{m,w} \geq 0$.

A matching μ is a ***core matching*** of assignment game α if

$$\mu \in \arg\max_{\nu \in \mathcal{M}} \sum_{m \in M} \sum_{w \in W} 1_{\nu(m)=w} \alpha_{m,w}.$$

For an assignment game α, denote the set of core matchings by $\mathcal{C}(\alpha)$. Our aim is to understand exactly which sets of matchings coincide with core matchings of some game α.

We proceed to describe a general model of coalition formation with transfers; two-sided assignment games are a special case of this model.

Let N be a set of agents; a ***characteristic function game*** is a function $v : 2^N \to \mathbf{R}$. A ***coalition structure*** over N is a partition of N. Let $\mathcal{P}$ be a family of partitions of N.[1]

We interpret $\mathcal{P}$ as the set of feasible coalitions. For example, in the assignment game $N = M \cup W$ and $\mathcal{P}$ corresponds to the partitions into pairs and singletons defined by some matching. Another example is the roommate game, where $\mathcal{P}$ corresponds to all partitions (S_i) of N with $|S_i| \leq 2$.

[1] Kaneko and Wooders (1982) is an early reference on this model.

If Π is a coalition structure over N, we associate with Π the ***value***

$$\sum_{S \in \Pi} v(S).$$

A partition $\Pi \in \mathcal{P}$ is ***optimal*** if its value is maximal in $\mathcal{P}$. Let $\mathcal{O}(v)$ denote the set of all optimal partitions for v.

3 The Results

3.1 General assignment games and consistency.

We will say that a set $E \subseteq \mathcal{M}$ is ***consistent*** if, whenever $\nu \in \mathcal{M}$ has the property that for all $i \in M \cup W$, there exists $\mu \in E$ for which $\nu(i) = \mu(i)$, then $\nu \in E$. We can rephrase the definition as follows. Say that a set E of matchings ***generates*** a matching $\nu \in \mathcal{M}$ if, for all $i \in M \cup W$, there is $\mu \in E$ with $\nu(i) = \mu(i)$. A set E is consistent if any matching that it generates is in E.

Theorem 1. *Let $E \subseteq \mathcal{M}$. The following statements are equivalent.*

i) There exists an integer valued assignment game α such that $E = \mathcal{C}(\alpha)$.

ii) There exists an assignment game α such that $E = \mathcal{C}(\alpha)$.

iii) The set E is consistent.

The proof of Theorem 1 is in Section 4. There is a simple constructive proof of the statement that iii) implies i) in the case where all matchings in E are complete. In that case, one can construct an assignment game by letting $\alpha_{ij} = 1$ if there is $\mu \in E$ with $\mu(i) = j$, and $\alpha_{ij} = 0$ otherwise. It is easy to verify that consistency implies $E = \mathcal{C}(\alpha)$ with this construction. There is also a simple proof that ii) implies iii) in the non-negative case (Section 3.2); the proof involves using Shapley and Shubik's (1971) theorem on the core payoffs.[2]

Our proof uses a different approach, based on Linear Programming techniques (different from the LP problem in Shapley and Shubik (1971)). Our proof reflects the method we used to discover Theorem 1. Once the result is know, simpler proofs may be feasible. We prefer to present the proof in Section 4 as a transparent guide to the results; in addition, it allows a unified treatment of Theorems 1, 4 and 6. As we explain in Section 3.4, it shows explicitly (by a version of the Birkhoff von-Neumann Theorem) where the two-sidedness of the assignment game matters.

Proposition 2. *For any set of matchings there is a unique smallest consistent set which contains it.*

Proof. First note that if E and E' are consistent sets of matchings, then $E \cap E'$ is consistent: Let $\nu \in \mathcal{M}$ have the property that for all i there exists $\mu \in E \cap E'$ for which $\nu(i) = \mu(i)$. Then, ν is generated by E and by E'. By consistency, $\nu \in E \cap E'$. The result follows because $\mathcal{M}$ is a consistent set of matchings, as the set of consistent supersets of E if nonempty and closed under intersections. ∎

Note that the smallest consistent set which contains E can be constructed by successively adding matchings that are generated by E.

Observe that by Proposition 2, we can test whether or not a set of observed matchings could be a *subset* of the set of core matchings of some assignment game. In particular, we may observe some matchings, but be unsure whether or not there are other matchings which could potentially observed. In order to test this hypothesis, there must be some set of matchings F which we know are *not* core matchings. Let E' be the smallest consistent

[2]Eran Shmaya pointed this out to us.

set containing E. Then there is a game α with $E \subseteq \mathcal{C}(\alpha)$ and $F \subseteq \mathcal{M} \setminus \mathcal{C}(\alpha)$ if and only if $F \subseteq \mathcal{M} \setminus E'$. This observation yields the following corollary:

Corollary 3. *Let E and F be nonempty disjoint sets of matchings. There is an assignment game α with $E \subseteq \mathcal{C}(\alpha)$ and $F \subseteq \mathcal{M} \setminus \mathcal{C}(\alpha)$, if and only if there is no $\nu \in F$ that is generated from E.*

As a simple application of Theorem 1, note that the matchings defined by the n cyclic permutations, with $n = |W| = |M|$, cannot be the core of an assignment game. Indeed, let $M = \{m_k : k = 1, \ldots, n\}$, $W = \{w_k : k = 1, \ldots, n\}$ and consider the set E of matchings defined by $\mu^k(w_i) = m_{i+k \bmod n}$, $k = 0, \ldots, n-1$. Let $\nu(w_i) = \mu_1(w_i)$ for all $i = 3, \ldots n$ and let $\nu(m_i) = w_{3-j}$, $j = 1, 2$. Then ν is generated by E, but ν is not a cyclic permutation.

3.2 Nonnegative assignment games and monotone consistency.

We may further ask whether there are additional conditions which are required on a set of matchings E to imply that E is the set of core matchings of an assignment game with nonnegative entries. Indeed such additional conditions exist. Observe that it is implicit in the definition of core matchings that single agents generate zero surplus: they do not contribute to the sum being optimized. It is then to be expected that the additional conditions on E involve single agents.

We say that a set of matchings E is ***monotone consistent*** if and only for all $\nu \in \mathcal{M}$, if for all $i \in M \cup W$, either there exists $\mu \in E$ for which $\mu(i) = \nu(i)$, or there exists μ, $\mu' \in E$ for which $\mu(i) = i$ and $\mu'(\nu(i)) = \nu(i)$, then $\nu \in E$.

Monotone consistency then requires that E not only contain the matchings which are generated from E, but also those matchings ν for which $\nu(i) \notin \{\mu(i) : \mu \in E\}$ for some i, as long as both i and $\nu(i)$ are single in some (possibly different) matchings in E.

Theorem 4. *Let $E \subseteq \mathcal{M}$. The following statements are equivalent.*

i) There exists a nonnegative integer valued assignment game α such that $E = \mathcal{C}(\alpha)$.

ii) There exists a nonnegative valued assignment game α such that $E = \mathcal{C}(\alpha)$.

iii) The set E is monotone consistent.

The proof of Theorem 4 is in Section 4.

Example 5. This simple example illustrates the difference between consistency and monotone consistency. Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Consider the matchings μ_1 and μ_2 with $\mu_1(m_k) = w_k$ for $k = 1, 2, 3$ and $\mu_2(m_k) = w_{k-1}$ for $k = 2, 3$ while $\mu_2(m_1) = w_3$ while $\mu_1(m_4) = \mu_2(m_4) = m_4$. Let $E = \{\mu_1, \mu_2\}$. There is no nonnegative assignment game α for which $E = \mathcal{C}(\alpha)$. On the other hand, E satisfies consistency. In order to obtain a monotone consistent set of matchings, we would need to add the matching $\nu(m_k) = w_k$ to E.

The statements in Proposition 2 and Corollary 3 corresponding to monotone consistency are true, and have very simple proofs.

3.3 General coalition formation with transfers.

We present a characterization for general coalition formation games. The result is simple and the characterization probably not surprising; its value lies in the contrast with the results on the assignment game. The characterization for assignment games (Sections 3.1 and 3.2) involves a stronger and more intuitive condition. We wish to emphasize how the two-sided structure of the assignment games makes an important difference here.

Let $(\Pi_i)_{i=1}^n$ and $(\Pi'_i)_{i=1}^n$ be sequences of partitions in $\mathcal{P}$. Say that $(\Pi'_i)_{i=1}^n$ is an ***arrangement*** of $(\Pi_i)_{i=1}^n$ if, for all S, the number of times S is a cell of some partition in $(\Pi_i)_{i=1}^n$ is the same as the number of times it is the cell of some partition in $(\Pi'_i)_{i=1}^n$.

The idea is that the partitions $(\Pi_i')_{i=1}^n$ are constructed using only cells from the partitions in $(\Pi_i)_{i=1}^n$, and such that a cell must be available in $(\Pi_i)_{i=1}^n$ as many times as it is used in $(\Pi_i')_{i=1}^n$. Thus, $(\Pi_i')_{i=1}^n$ is an arrangement of $(\Pi_i)_{i=1}^n$ if for all Π_i of which S is a cell, there is a distinct Π_i' of which S is a cell; and vice versa for all S, for all Π_i' of which S is a cell, there is a distinct Π_i of which S is a cell.

For example, with $N = \{1, 2, 3, 4, 5\}$, consider the following partitions.

$$\Pi_1 : \{1,2\}\,\{3,4\}\,,\{5\}$$

$$\Pi_2 : \{1\}\,\{2,5\}\,,\{3\}\,,\{4\}$$

$$\Pi_3 : \{1,2,3\}\,\{4,5\}$$

$$\Pi_1' : \{1,2\}\,,\{3\}\,,\{4,5\}$$

$$\Pi_2' : \{1\}\,,\{2,5\}\,,\{3,4\}$$

$$\Pi_3' : \{1,2,3\}\,,\{4\}\,,\{5\}$$

Note how (Π_1', Π_2', Π_3') is an arrangement of (Π_1, Π_2, Π_3).

A set of partitions $E \subseteq \mathcal{P}$ is ***closed under arrangements*** if, whenever $(\Pi_i')_{i=1}^n$ is an arrangement of partitions in E, we have $\Pi_i' \in E$, $i = 1, \ldots, n$.

Theorem 6. *Let $E \subseteq \mathcal{P}$. The following statements are equivalent.*

i) There exists an integer valued characteristic function game $v : 2^N \to \mathbb{Z}$ such that $E = \mathcal{O}(v)$.

ii) There exists a characteristic function game $v : 2^N \to \mathbf{R}$ such that $E = \mathcal{O}(v)$.

iii) The set E is closed under arrangements.

The proof of Theorem 6 is in Section 4. The proof is simple. Optimality involves maximizing a sum of values which only depend on the cells of the partition. Hence, an arrangement must provide the same value as any sum of maximizing partitions.

The result in Theorem 6 is not surprising. In assignment games, though, the two-sided structure of the problem provides a stronger characterization.

3.4 Assignment games and general coalition-formation games.

The two-sided nature of assignment games is responsible for the stronger results in theorems 1 and 4. The following is the crucial consequence of two-sidedness (for our purposes).

In general coalition-formation games, one may "generate" a partition from some sequence $(\Pi_i)_{i=1}^n$ in a way that the remaining coalitions cannot be re-arranged into $n-1$ partitions. For example, consider a roommate model with $N = \{1, 2, 3\}$ and the partitions

$$\Pi_1 : \{1\}\,\{2,3\}$$

$$\Pi_2 : \{2\}\,\{1,3\}$$

$$\Pi_3 : \{3\}\,\{1,2\}\,;$$

these generate (in the obvious sense) the partition into singletons: $\{1\}$, $\{2\}$, $\{3\}$. But the remaining cells, $\{2,3\}$, $\{1,2\}$, $\{1,3\}$, cannot be arranged into a collection of partitions.

In assignment games, this situation cannot arise. If we generate a matching from n matchings, then the remaining pairs can always be collected into $n-1$ matchings; this is the main thrust of the proof of Theorem 1.

For example, if $E = \{\mu_1, \mu_2, \mu_3\}$ generates $\nu \notin E$, then Step 1 in the proof of Theorem 1 guarantees that, with the pairs (and singletons) that are left after generating ν, we can always generate two matchings ν' and ν''. Since surpluses only depend on individual pairs, the sum of payoffs in all matchings in $\{\nu, nu', \nu''\}$ has to equal the sum of payoffs in all matchings in E. This contradicts that E is the set of core matchings.

3.5 Assignment games and matching markets with no transfers

We present some examples to clarify the relationship between assignment games and matching markets without transfers.

First, we show that there are sets of matchings E which can be the core of one model but not the other. One might initially believe that the model with transfers should have more predictive power than the model without. This turns out to be false; our first example shows that the stable matchings for a model without transfers may never be the core of an assignment game. Second, we present a consistent set of matchings that cannot be stable, for any preferences in the model without transfers.

The following is a succinct description of the model without transfers (Gale and Shapley, 1962; Roth and Sotomayor, 1990): Let M and W be finite, disjoint, sets. For $m \in M$, a ***preference***, $P(m)$, is a linear order over $W \cup \{m\}$. For $w \in W$, a ***preference***, $P(w)$, is a linear order over $M \cup \{w\}$.

Given lists of preferences $(P(m))_{m\in M}$ and $(P(w))_{w\in W}$, a matching μ is ***stable*** if,

i) for all $i \in M \cup W$ with $\mu(i) \neq i$, $\mu(i)\, P(i)\, i$;

ii) there is no $(m, w) \in M \times W$ with $w \neq \mu(m)$ and $w\, P(m)\, \mu(m)$ and $m\, P(w)\, \mu(w)$.

Example 7. This example describes an inconsistent set E which is nevertheless a set of stable matchings. Hence there are sets of stable matchings which cannot be the core of an assignment game. Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Consider the preferences:

$$
\begin{array}{lllll@{\qquad}lllll}
P(m_1): & w_1 & w_2 & w_3 & w_4 & P(w_1): & m_4 & m_3 & m_2 & m_1 \\
P(m_2): & w_2 & w_1 & w_4 & w_3 & P(w_2): & m_3 & m_4 & m_1 & m_2 \\
P(m_3): & w_3 & w_4 & w_1 & w_2 & P(w_3): & m_2 & m_1 & m_4 & m_3 \\
P(m_4): & w_4 & w_3 & w_2 & w_1 & P(w_4): & m_1 & m_2 & m_3 & m_4
\end{array}
$$

Then the matchings

	m_1	m_2	m_3	m_4
$\mu_1:$	w_2	w_4	w_1	w_3
$\mu_2:$	w_2	w_1	w_3	w_4
$\mu_3:$	w_3	w_4	w_2	w_1

are stable.[3] The table means that where $\mu_1(m_1) = w_2$, $\mu_3(m_1) = w_3$, and so on. The matching

	m_1	m_2	m_3	m_4
$\nu:$	w_2	w_4	w_3	w_1

is generated from $\{\mu_1, \mu_2, \mu_3\}$ but it is not stable, as $w_2 \, P(m_4) \, w_1 = \nu(m_4)$ and $m_4 \, P(w_2)$ $m_1 = \nu(w_2)$.

There are more stable matchings than those in E, but since ν is generated by E, it is generated by the set of stable matchings. Since ν is unstable, the set of stable matchings is inconsistent.

Example 8. Our second example is of a consistent set of matchings that cannot be stable under any preference profile. Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$. Consider the set $E = \{\mu_1, \mu_2, \mu_3\}$ of matchings described as follows:

	m_1	m_2	m_3	m_4
$\mu_1:$	w_1	w_2	w_3	w_4
$\mu_2:$	w_1	w_3	w_4	w_2
$\mu_3:$	w_2	w_3	w_1	$w_4,$

The set E is rationalizable as the core of an assignment game, but not as the core of a marriage matching model. We show that E is consistent. If ν is a matching generated by E we must have $\nu(m_1) \in \{w_1, w_2\}$. Say that $\nu(m_1) = w_1$. We must have $\nu(m_2) \in \{w_2, w_3\}$. If $\nu(m_2) = w_2$, then $\nu(m_4) = w_4$ and so $\nu = \mu_1$. If $\nu(m_2) = w_3$ then $\nu(m_3) = w_4$

[3] Our example is taken from the example in Figures 1.9 and 1.10 in Gusfield and Irving (1989)

(as $\nu(m_1) = w_1$) so $\nu(m_4) = w_2$ and $\nu = \mu_2$. On the other hand, if $\nu(m_1) = w_2$ we must have $\nu(m_2) = w_3$ and $\nu(m_4) = w_4$. Hence $\nu(m_3) = w_1$ and $\nu = \mu_3$.

On the other hand, the matchings in E cannot be the set of stable matchings of a non-transferable-utility marriage market. Suppose, by way of contradiction, that $(P(m))_{m\in M}$ and $(P(w))_{w\in W}$ are preference profiles such that the matchings in E are all stable (admitting that more matchings than those in E might be stable). Say that $w_2\, P(m_2)\, w_3$. This rules out the possibility that $w_2\, P(m_4)\, w_4$, as μ_2 would then be unstable if $m_2\, P(w_2)\, m_4$ and μ_1 would be unstable if $m_4\, P(w_2)\, m_2$. So we must have that $w_4\, P(m_4)\, w_2$. In turn, this implies that $w_3\, P(m_3)\, w_4$ by a similar argument. Now, the stability of μ_1 and μ_3 and $w_2\, P(m_2)\, w_3$ implies, by the same argument as above, that $w_1\, P(m_1)\, w_2$. Then we obtain that $w_3\, P(m_3)\, w_1$. Finally, $w_1\, P(m_1)\, w_2$ and the stability of μ_2 and μ_3 obtains that $w_4\, P(m_3)\, w_1$. But we established that $w_4\, P(m_4)\, w_2$, so if $m_3\, P(w_4)\, m_4$ μ_3 is unstable, and if $m_4\, P(w_4)\, m_3$ then μ_2 is unstable.

Note that we obtain the same conclusion if we assume instead that $w_3\, P(m_2)\, w_2$.

The previous example is particularly interesting because all matchings in E are complete. There are simpler examples based on the property that any two stable matchings must have the same set of single agents. For example, with two men and two women, consider the matchings defined by $\mu_1(m_1) = \mu_2(m_2) = w_1$ and $\mu_1(m_2) = m_2$ and $\mu_2(m_1) = m_1$. This set is evidently consistent, but the two matchings could not be stable.

4 Proofs

We start with the following lemma, whose proof was shown to us by Kim Border. [4]

[4] Kim Border claims the result is well-known, but we were unable to find a reference. The lemma is a simple consequence of the standard Farkas's Lemma and of the rational version of Farkas's Lemma (see

Lemma 9. *(Integer-Real Farkas) Let* $\{A_i\}_{i=1}^K$ *be a finite collection of vectors in* $\mathbb{Q}^n$*. Then one and only one of the following statements is true:*

i) There exists $y \in \mathbb{R}^n$ *such that for all* $i = 1, ..., L,$ $A_i \cdot y \geq 0$ *and for all* $i = L+1, ..., K,$ $A_i \cdot y > 0$.

ii) There exists $z \in \mathbb{Z}_+^K$ *such that* $\sum_{i=1}^K z_i A_i = 0$*, where* $\sum_{i=L+1}^K z_i > 0$.

Proof. It is clear that both i) and ii) cannot simultaneously hold. We therefore establish that if ii) does not hold, i) holds. By Theorem 3.2 of Fishburn (1973), if ii) does not hold, there exists $q \in \mathbb{Q}^n$ such that for all $i = 1, ..., L,$ $A_i \cdot q \geq 0$ and for all $i = L+1, ..., K,$ $A_i \cdot q > 0$. Hence, $q \in \mathbb{R}^n$. ∎

Lemma 10. *Let* $\{A_i\}_{i=1}^K$ *be a collection of vectors in* $\mathbb{Q}^n$*. Then there exists* $y \in \mathbb{R}^n$ *such that for all* $i = 1, ..., L,$ $A_i \cdot y \geq 0$ *and for all* $i = L+1, ..., K,$ $A_i \cdot y > 0$ *if and only if there exists* $z \in \mathbb{Z}^n$ *such that for all* $i = 1, ..., L,$ $A_i \cdot z \geq 0$ *and for all* $i = L+1, ..., K,$ $A_i \cdot z > 0$.

Proof. Immediate from Theorem 3.2 of Fishburn (1973) and Lemma 9. ∎

4.1 Proof of Theorem 1

We first establish the equivalence of i) and ii). The existence of an assignment game α is equivalent to the existence of $\alpha \in \mathbb{R}^{M \times W}$ for which for all $\mu \in E$ and all $\nu \in \mathcal{M}$,

$$\sum_{m \in M} \sum_{w \in W} \left(1_{\mu(m)=w} - 1_{\nu(m)=w}\right) \alpha_{m,w} \geq 0, \tag{1}$$

Gale (1960) and Fishburn (1973) or Fishburn (1971)). It is crucial since it allows one to relate a primal involving real numbers with a dual involving integers.

and for all $\mu \in E$ and all $\nu \notin E$

$$\sum_{m\in M}\sum_{w\in W}\left(1_{\mu(m)=w}-1_{\nu(m)=w}\right)\alpha_{m,w}>0. \tag{2}$$

As each of the vectors $\left(1_{\mu(m)=w}-1_{\nu(m)=w}\right)_{m,w}$ are rational valued, the claim follows from Lemma 10.

Now, we establish the equivalence of ii) and iii).

Step 1: A characterization of sums of matrices associated with matchings, using Hall's Theorem.

The result in this step is closely related to the well-known Birkhoff-von Neumann Theorem, but is distinct from this result. Let $\mu \in \mathcal{M}$ be a matching. Associated with this matching is the matrix

$$\left(1_{\mu(m)=w}\right)_{m,w}.$$

Note that by definition, for all $w \in W$,

$$\sum_{m\in M} 1_{\mu(m)=w} \leq 1;$$

this follows as if $\mu(m') = \mu(m) = w$, then $m = \mu(\mu(m)) = \mu(w) = \mu(\mu(m')) = m'$. Likewise, for all $m \in M$,

$$\sum_{w\in W} 1_{\mu(m)=w} \leq 1;$$

this follows simply as μ as a function. Consequently, if $\{\mu_1, ..., \mu_n\}$ is a finite list of matchings, then for all $w \in W$,

$$\sum_{i=1}^{n}\sum_{m\in M} 1_{\mu_i(m)=w} \leq n$$

and for all $m \in M$,

$$\sum_{i=1}^{n} \sum_{w \in W} 1_{\mu_i(m)=w} \leq n.$$

Conversely, suppose that $A \in \mathbb{Z}_+^{M \times W}$ satisfies for all $w \in W$,

$$\sum_{m \in M} A_{m,w} \leq n$$

and for all $m \in M$,

$$\sum_{w \in W} A_{m,w} \leq n,$$

then there exists a list of matchings $\{\mu_1, ..., \mu_n\}$ for which for all $(m, w) \in M \times W$,

$$A_{m,w} = \sum_{i=1}^{n} 1_{\mu_i(m)=w}.$$

To see this, we construct a matrix $A' \in \mathbb{Z}_+^{(M \cup W) \times (M \cup W)}$ defined so that for all $m \in M$, $w \in W$, $A'_{m,w} = A'_{w,m} = A_{m,w}$, for all $m, m' \in M$ for which $m \neq m'$, $A'_{m,m'} = 0$, for all $w, w' \in W$ for which $w \neq w'$, $A_{w,w'} = 0$, for all $m \in M$, $A'_{m,m} = n - \sum_{w \in W} A_{m,w}$, and for all $w \in W$, $A'_{w,w} = n - \sum_{m \in M} A_{m,w}$. Note in particular that the matrix A' has the property that for all $x \in M \cup W$

$$\sum_{y \in M \cup W} A'_{x,y} = \sum_{y \in M \cup W} A'_{y,x} = n.$$

Now, consider the correspondence $\Gamma : M \cup W \rightrightarrows M \cup W$ defined by

$$\Gamma(x) = \left\{ y : A'_{x,y} > 0 \right\}.$$

We first show that there exists a function $\gamma : M \cup W \to M \cup W$ for which i) for all $x \in M \cup W$, $\gamma(x) \in \Gamma(x)$ and ii) for all $x, x' \in M$ for which $x \neq x'$, $\gamma(x) \neq \gamma(x')$. To do so, we will use the Theorem of König and Hall, which states that the existence of such a γ will follow if we can establish that for all $F \subseteq M \cup W$, $\left| \bigcup_{x \in F} \Gamma(x) \right| \geq |F|$(see *e.g.* Berge (2001), Chapter 10).

We proceed by induction on the cardinality of F. If $|F| = 1$, then the result is trivial: let $F = \{x\}$, then as $\sum_{y \in M \cup W} A'_{x,y} = n$, there exists y for which $A'_{x,y} > 0$.

Now suppose the statement is true for all $F \subseteq M$ for which $|F| \leq k-1$, and let $F' \subseteq M$ have cardinality $|F'| = k$. We shall prove that $\left| \bigcup_{x \in F'} \Gamma(x) \right| \geq k$. Fix $x' \in F'$; note that by the induction hypothesis

$$\left| \bigcup_{x \in F' \backslash \{x'\}} \Gamma(x) \right| \geq k-1.$$

If in fact $\left| \bigcup_{x \in F' \backslash \{x'\}} \Gamma(x) \right| > k-1$, then as $\left| \bigcup_{x \in F'} \Gamma(x) \right| \geq \left| \bigcup_{x \in F' \backslash \{x'\}} \Gamma(x) \right| \geq k$, we have established the claim. So suppose that $\left| \bigcup_{x \in F' \backslash \{x'\}} \Gamma(x) \right| = k-1$.

As for all $x \in F' \backslash \{x'\}$ and all B such that $\Gamma(x) \subseteq B$,

$$\sum_{y \in B} A'_{x,y} = n,$$

we obtain

$$\sum_{x \in F' \backslash \{x'\}} \left[\sum_{y \in \bigcup_{\tilde{x} \in F' \backslash \{x'\}} \Gamma(\tilde{x})} A'_{x,y} \right] = n(k-1). \tag{3}$$

On the other hand, for all $y \in \bigcup_{\tilde{x} \in F' \backslash \{x'\}} \Gamma(\tilde{x})$, $\sum_{x \in M \cup W} A'_{x,y} = n$. Hence,

$\left|\bigcup_{x\in F'\setminus\{x'\}}\Gamma(x)\right| = k-1$ implies that

$$\sum_{y\in\ \bigcup_{\tilde{x}\in F'\setminus\{x'\}}\Gamma(\tilde{x})}\left[\sum_{x\in M\cup W}A'_{x,y}\right] = n\,(k-1)\,.$$

Reversing sums in the latter equality, and using (3), obtains

$$\sum_{x\in M\cup W}\left[\sum_{y\in\ \bigcup_{\tilde{x}\in F'\setminus\{x'\}}\Gamma(\tilde{x})}A'_{x,y}\right] = n\,(k-1) = \sum_{x\in F'\setminus\{x'\}}\left[\sum_{y\in\ \bigcup_{\tilde{x}\in F'\setminus\{x'\}}\Gamma(\tilde{x})}A'_{x,y}\right].$$

Consequently,

$$\sum_{y\in\ \bigcup_{\tilde{x}\in F'\setminus\{x'\}}\Gamma(\tilde{x})}A'_{x',y} \leq \sum_{x\notin(F'\setminus\{x'\})}\left[\sum_{y\in\ \bigcup_{\tilde{x}\in F'\setminus\{x'\}}\Gamma(\tilde{x})}A'_{x,y}\right] = 0.$$

Hence $A'_{x',y} = 0$ for all $y \in \bigcup_{\tilde{x}\in F'\setminus\{x'\}}\Gamma(\tilde{x})$. Conclude that there exists $y \notin \bigcup_{\tilde{x}\in F'\setminus\{x'\}}\Gamma(\tilde{x})$ for which $A_{x',y} > 0$, so that $\Gamma(x') \subseteq \bigcup_{\tilde{x}\in F'\setminus\{x'\}}\Gamma(\tilde{x})$ is false. Hence $\left|\bigcup_{x\in F'}\Gamma(x)\right| > \left|\bigcup_{x\in F'\setminus\{x'\}}\Gamma(x)\right| = k-1$, so that $\left|\bigcup_{x\in F'}\Gamma(x)\right| \geq k = |F'|$, verifying the claim.

Let γ be the aforementioned mapping. Importantly, $\gamma(M\cup W) = M\cup W$. Now, for all $m\in M$, define $\mu(m) = \gamma(m)$. For all $w\in W$, if $w = \mu(m)$ for some $m\in M$, define $\mu(w) = m$. Otherwise, define $\mu(w) = w$. To see that μ is a matching, note that we only need to verify that $\mu(m) \notin M\setminus\{m\}$. Suppose by means of contradiction that $\mu(m) = m' \in M\setminus\{m\}$. Then in particular, $\mu(m)\in\Gamma(m)$, which implies that $A'_{m,m'} > 0$, a contradiction.

We finish the proof of Step 1 by induction. We show that μ induces a matrix $B \in \mathbb{Z}_+^{M\times W}$ such that $A - B$ is a nonnegative integer valued matrix and for all $m\in M$ and

$w \in W$:

$$\sum_{\tilde{w} \in W} (A_{m,\tilde{w}} - B_{m,\tilde{w}}) \leq n - 1 \tag{4}$$

$$\sum_{\tilde{m} \in M} (A_{\tilde{m},w} - B_{\tilde{m},w}) \leq n - 1; \tag{5}$$

thus $A - B$ is under the hypotheses that allowed us to define the matching μ above. By applying the argument inductively, we show that A defines a collection of matchings, as stated in Step 1.

First, let $B \in \mathbb{Z}_+^{M \times W}$ be the matrix $[B_{m,w}] = 1_{\mu(m)=w}$. We claim that $B \leq A$; so let $(m, w) \in M \times W$ be arbitrary. If $B_{m,w} = 0$ then $B_{m,w} \leq A_{m,w}$ by assumption on A. If $B_{m,w} = 1$, then $\mu(m) = w$; hence $\gamma(m) = w$ and $A_{m,w} = A'_{m,w} > 0$. Consequently, $B_{m,w} \leq A_{m,w}$. This proves that $A - B$ is nonnegative.

Second, we show that (4) holds by showing that, for all $m \in M$, if $\sum_{w \in W} A_{m,w} = n$, then $\sum_{w \in W} 1_{\mu(m)=w} = 1$. This follows as if $\sum_{m \in M} A_{m,w} = n$, then $A_{m,m} = 0$, so that $\Gamma(m) \subseteq W$, consequently, $\mu(m) = \gamma(m) \in W$. So $B_{m,\gamma(m)} = 1$. Lastly, we show (5) by showing that for all $w \in W$, if $\sum_{m \in M} A_{m,w} = n$, then $\sum_{m \in M} 1_{\mu(m)=w} = 1$. So suppose that $\sum_{m \in M} A_{m,w} = n$. Then $A_{w,w} = 0$. As $\gamma(M \cup W) = M \cup W$, there exists some $x \in M \cup W$ for which $\gamma(x) = w$. But it as $A_{w,w} = 0$, $\gamma(w) = w$ is impossible. Conclude that there exists some $m \in M$ for which $\gamma(m) = w$, or $\mu(m) = w$; hence $B_{m,w} = 1$.

Step 2: A characterization of non-existence of a rationalizing assignment game using the Integer-Real Farkas Lemma.

We will show that the converses of ii) and iii) are equivalent. The existence of an assignment game α for which $E = C(\alpha)$ is equivalent to the existence of α for which for all $\mu \in E$ and all $\nu \in \mathcal{M}$,

$$\sum_{m \in M} \sum_{w \in W} \left(1_{\mu(m)=w} - 1_{\nu(m)=w}\right) \alpha_{m,w} \geq 0,$$

and for all $\mu \in E$ and all $\nu \notin E$

$$\sum_{m \in M} \sum_{w \in W} \left(1_{\mu(m)=w} - 1_{\nu(m)=w}\right) \alpha_{m,w} > 0.$$

Hence, the nonexistence of such an α is equivalent, by the Lemma 9, to the existence of a vector $z \in \mathbb{Z}_+^{E \times \mathcal{M}}$ such that for some $(\mu, \nu) \in E \times (\mathcal{M} \backslash E)$, $z_{\mu,\nu} > 0$, and for all $(m, w) \in M \times W$,

$$\sum_{(\mu,\nu) \in E \times \mathcal{M}} z_{\mu,\nu} \left(1_{\mu(m)=w} - 1_{\nu(m)=w}\right) = 0.$$

Step 3: Some basic algebraic manipulation.

The non-existence of α with the above properties is equivalent to the existence of a finite list of matchings $\{\mu_1, ..., \mu_n\} \subseteq E$, and a finite list of matchings $\{\nu_1, ..., \nu_n\} \subseteq \mathcal{M}$ such that there exists $j \in \{1, ..., n\}$ for which $\nu_j \in \mathcal{M} \backslash E$, such that for all $(m, w) \in M \times W$

$$\sum_{i=1}^{n} \left(1_{\mu_i(m)=w}\right) = \sum_{i=1}^{n} \left(1_{\nu_i(m)=w}\right).$$

Suppose without loss of generality that $\nu_n \in \mathcal{M} \backslash E$; we rewrite the preceding as for all $(m, w) \in M \times W$,

$$\left[\sum_{i=1}^{n} \left(1_{\mu_i(m)=w}\right)\right] - \left(1_{\nu_n(m)=w}\right) = \sum_{i=1}^{n-1} \left(1_{\nu_i(m)=w}\right). \tag{6}$$

The sum on the right of Equality (6) equals a sum of $n-1$ matchings. By Step 1, then, the existence of the two sets of matchings satisfying Equality (6) is equivalent to the existence of a finite list of matchings $\{\mu_1, ..., \mu_n\} \in E$ and a matching $\nu \in \mathcal{M} \setminus E$ such that for all $m \in M$

$$\sum_{w \in W} \left[\left[\sum_{i=1}^{n} \left(1_{\mu_i(m)=w} \right) \right] - \left(1_{\nu(m)=w} \right) \right] \leq n - 1$$

and all $w \in W$

$$\sum_{m \in M} \left[\left[\sum_{i=1}^{n} \left(1_{\mu_i(m)=w} \right) \right] - \left(1_{\nu(m)=w} \right) \right] \leq n - 1,$$

and all pairs $(m, w) \in M \times W$, $\left[\sum_{i=1}^{n} \left(1_{\mu_i(m)=w} \right) \right] - \left(1_{\nu(m)=w} \right) \geq 0$.

The first inequality is satisfied if and only if whenever $\nu(m) = m$, there exists i for which $\mu_i(m) = m$. The second inequality is satisfied if and only if whenever $\nu(w) = w$, there exists i for which $\mu_i(w) = w$. The last inequality is satisfied if and only if whenever $\nu(m) = w$, there exists i for which $\mu_i(m) = w$. The existence of such matchings therefore occurs if and only if E is not consistent.

4.2 Proof of Theorem 4

That i) and ii) are equivalent follow similarly to Theorem 1. For the equivalence of ii) and iii), note that the existence of a nonnegative valued assignment game α for which $E = C(\alpha)$ is equivalent to the existence of $\alpha \in \mathbb{R}_{+}^{M \times W}$ for which for all $\mu \in E$ and all $\nu \in \mathcal{M}$,

$$\sum_{m \in M} \sum_{w \in W} \left(1_{\mu(m)=w} - 1_{\nu(m)=w} \right) \alpha_{m,w} \geq 0,$$

for all $\mu \in E$ and all $\nu \notin E$

$$\sum_{m \in M} \sum_{w \in W} \left(1_{\mu(m)=w} - 1_{\nu(m)=w} \right) \alpha_{m,w} > 0,$$

and for all $(m,w) \in M \times W$,

$$1_{m,w}\alpha_{m,w} \geq 0.$$

Hence, the nonexistence of such an α is equivalent, by Lemma 9, to the existence of a vector $z \in \mathbb{Z}_+^{E\times\mathcal{M}}$ such that for some $(\mu,\nu) \in E \times (\mathcal{M}\backslash E)$, $z_{\mu,\nu} > 0$, and a vector $z' \in \mathbb{Z}_+^{M\times W}$ for which for all $(m,w) \in M \times W$,

$$\sum_{(\mu,\nu)\in E\times\mathcal{M}} z_{\mu,\nu}\left[1_{\mu(m)=w} - 1_{\nu(m)=w}\right] + z'_{m,w} = 0.$$

As in the proof of Theorem 1, this is equivalent to the existence of a finite list of matchings $\{\mu_1,...,\mu_n\} \subseteq E$, a finite list of matchings $\{\nu_1,...,\nu_n\} \subseteq \mathcal{M}$ such that there exists $j \in \{1,...,n\}$ for which $\nu_j \in \mathcal{M}\backslash E$, and for all $(m,w) \in M \times W$ an integer $z_{m,w} \geq 0$ for which for all $(m,w) \in M \times W$,

$$\left[\sum_{i=1}^{n} 1_{\mu_i(m)=w}\right] = \left[\sum_{i=1}^{n} 1_{\nu_i(m)=w} - z_{m,w}\right].$$

Suppose without loss of generality that ν_n is not an element of E; let $\nu = \nu_n$. Therefore the previous equality is equivalent to the existence of matchings $\{\mu_1,...,\mu_n\} \subseteq E$, a matching $\nu \in \mathcal{M}\backslash E$, and $z_{m,w} \geq 0$ for all $(m,w) \in M \times W$ for which for all $(m,w) \in M \times W$

$$\left[\sum_{i=1}^{n} 1_{\mu_i(m)=w} - 1_{\nu(m)=w}\right] = \left[\sum_{i=1}^{n-1} 1_{\nu_i(m)=w} - z_{m,w}\right]. \tag{7}$$

For $x \in \mathbb{R}$, define $x^+ = \max\{0,x\}$.

The right hand side of (7) satisfies that, for all $m \in M$,

$$\sum_{w\in W}\left[\sum_{i=1}^{n-1} 1_{\nu_i(m)=w} - z_{m,w}\right]^+ \leq n-1$$

and for all $w \in W$, $\sum_{m\in M}\sum_{i=1}^{n-1}\left[1_{\nu_i(m)=w} - z_{m,w}\right]^+ \leq n-1$. In contrast to Theorem 1, the values of the matrix on the right-hand side of (7) may be negative. However, analogously to the proof of Theorem 1, it can be shown that if a matrix $A \in \mathbb{Z}^{M\times W}$ satisfies for all $m \in M$,

$$\sum_{w\in W} A^{+}_{m,w} \leq n-1$$

and for all $w \in W$,

$$\sum_{m\in M} A^{+}_{m,w} \leq n-1,$$

then there exist matchings $\{\nu_1, ..., \nu_{n-1}\} \subseteq \mathcal{M}$ and a vector $z \in \mathbb{Z}^{M\times W}_{+}$ for which for all $(m, w) \in M \times W$,

$$[A_{m,w}] = \left[\sum_{i=1}^{n-1} 1_{\nu_i(m)=w} - z_{m,w}\right].$$

This follows from the observation that $[A_{m,w}] = \left[A^{+}_{m,w}\right] + \left[A^{-}_{m,w}\right]$. Consequently, the non-existence of α under our condition, is equivalent to the existence of a collection of matchings $\{\mu_1, ..., \mu_n\} \subseteq E$ and $\nu \in \mathcal{M}\backslash E$ such that for all $w \in W$,

$$\sum_{m\in M}\left[\left(\left[\sum_{i=1}^{n} 1_{\mu_i(m)=w}\right] - 1_{\nu(m)=w}\right)^{+}\right] \leq n-1$$

and for all $m \in M$,

$$\sum_{w\in W}\left[\left(\left[\sum_{i=1}^{n} 1_{\mu_i(m)=w}\right] - 1_{\nu(m)=w}\right)^{+}\right] \leq n-1.$$

We claim that the first inequality is satisfied if and only if, for all $w \in W$:

1. if $\nu(w) = w$ then there is $i \in \{1, \ldots, n\}$ with $\mu_i(w) = w$.

2. if $\nu(w) \neq w$, then either there exists $i \in \{1, \ldots, n\}$ with $\mu_i(\nu(w)) = w$, or there is $i \in \{1, \ldots, n\}$ with $\mu_i(w) = \nu(w)$.

Consider case (1): note that $\nu(w) = w$ implies that for all m, $1_{\nu(m)=w} = 0$. Hence the first inequality is equivalent to $\sum_m \sum_i 1_{\mu_i(m)=w} \leq n-1$. This is true iff there is μ_i in which w is single.

For case (2), suppose that $\nu(w) = \hat{m} \neq w$, and that for all i, $\mu_i(w) \in M$. Then $\sum_m \sum_i 1_{\mu_i(m)=w} = n$. So for the first inequality to hold there must be some i with $1_{\mu_i(m)=w} - 1_{\nu(m)=w} = 0$; that is $\mu_i(w) = \nu(w)$.

Similarly, the second inequality is satisfied if and only if for all $m \in M$, if $\nu(m) = m$, then there exists i for which $\mu_i(m) = m$, and if $\nu(m) = w$, either there exists $i \in M$ for which $\mu_i(m) = w$ or there exists $i \in M$ for which $\mu_i(m) = m$.

But the existence of such matchings is equivalent to a violation of monotone consistency.

4.3 Proof of Theorem 6

That i) and ii) are equivalent follow similarly to Theorem 1.

Let $\Pi \in \mathcal{P}$. We identify Π with the vector $1^\Pi \in \{0,1\}^{2^N}$ defined by $1^\Pi_S = 1$ if and only if $S \in \Pi$. We can also identify any characteristic function game v with a vector $\bar{v} \in \mathbf{R}^{2^n}$; the property that Π is optimal in $\mathcal{P}$ is then expressed as, for all $\Pi' \in \mathcal{P}$, $1^\Pi \cdot \bar{v} \geq 1^{\Pi'} \cdot \bar{v}$.

Now, $E = \mathcal{O}(v)$ iff $1^\Pi \cdot \bar{v} \geq 1^{\Pi'} \cdot \bar{v}$ for all $\Pi \in E$ and $\Pi \in \mathcal{P}$; and $1^\Pi \cdot \bar{v} > 1^{\Pi'} \cdot \bar{v}$ for all $\Pi \in E$ and $\Pi \in \mathcal{P} \setminus E$. So the property $E = \mathcal{O}(v)$ is equivalent to $\bar{v}$ being a solution to the system of inequalities defined above; in this system, there is a weak inequality associated with each pair $(\Pi, \Pi') \in E \times \mathcal{P}$, and a strict inequality associated with each pair $(\Pi, \Pi') \in E \times \mathcal{P} \setminus E$.

There is a solution to the system iff there is are no collections of non-negative integers $(z_{(\Pi,\Pi')})_{(\Pi,\Pi') \in E \times \mathcal{P}}$ and $(z'_{(\Pi,\Pi')})_{(\Pi,\Pi') \in E \times \mathcal{P} \setminus E}$, with at least one of the latter being strictly positive, s.t.

$$\sum_{(\Pi,\Pi')\in E\times\mathcal{P}} z_{(\Pi,\Pi')}(1^{\Pi}-1^{\Pi'}) + \sum_{(\Pi,\Pi')\in E\times\mathcal{P}\setminus E} z'_{(\Pi,\Pi')}(1^{\Pi}-1^{\Pi'}) = 0$$

The collections $(z_{(\Pi,\Pi')})_{(\Pi,\Pi')\in E\times\mathcal{P}}$ and $(z'_{(\Pi,\Pi')})_{(\Pi,\Pi')\in E\times\mathcal{P}\setminus E}$ define a sequence Π_i, $i = 1\dots n$ in E and a sequence Π'_i $i = 1\dots n$ in $\mathcal{P}$, with at least one $\Pi'_i \in \mathcal{P}\setminus E$, with the property that

$$\sum_{i=1}^{n} 1^{\Pi_i} = \sum_{i=1}^{n} 1^{\Pi'_i}. \tag{8}$$

Property (8) says that (Π'_i) is an arrangement of (Π_i): The number of times a set $S \subseteq N$ appears as a cell of some Π_i is the same as the number it appears as a cell of some Π'_i.

5 Econometric Implementations

One can use our results to test the assignment game with realistic data. For example, consider data on a two-sided matching market where one identifies agents with the same observable characteristics: the standard procedure in the empirical literature on matching (Pollak, 1990; Choo and Siow, 2006; Dagsvik, 2000). This procedure gives a collection of matchings E between a given set of agents.

It is natural to ask, as we have done here, if a constant α can rationalize all the matchings in E. But in an empirical exercise one usually wants to allow for some unobserved heterogeneity (in matching markets, see for example Chiappori, Fortin, and Lacroix (2002), Heckman, Matzkin, and Nesheim (2005), Hitsch, Hortaçsu, and Ariely (2006) and Fox (2007)).

Suppose, then, that we want $\mu^k \in E$ be rationalized by some matrix

$$\alpha^k = \left[\alpha_{ij} + \epsilon_i^k + \epsilon_j^k\right],$$

where ϵ_i and ϵ_j are individual error terms drawn from some known probability distribution. It should be clear that our results are still useful for this exercise. We proceed to describe two possible tests.

First, if E is consistent, the set of rationalizing α is the set of solutions to a system of linear inequalities ((1),(2)): these are easy to solve computationally and give bounds on the location of individual α^k. So E *can* be rationalized by multiple α^k; these must lie in the closed convex set of solutions to ((1),(2)).

If E is not consistent one can consider the maximal consistent subsets E': each of these is associated with a set of solutions and one can compute the p-value of errors (ϵ_i^k) and (ϵ_j^k) comparing the value of the matchings in E' with the matchings in $E \setminus E'$. This gives a simple way of testing for core matchings while allowing for unobserved heterogeneity.

Second, one can decide to use Corollary 3: decide on a set of necessarily inefficient matchings and check for consistency. The linear programming formulation ((1),(2)) provides a set of possible rationalizations. One can use these to obtain bounds on α and test for core matchings as described.

6 Conclusion

This work has studied the structure of the set of core matchings of assignment games. This structure is relevant as in many real-world scenarios, transfers may not be observed, but the actual matchings are. We discuss some related questions which may be analyzed using similar techniques.

First is the question of assortative matchings (Becker (1973)). Becker establishes that, when men and women have equal cardinalities and each set is linearly ordered, if the resulting function α is strictly supermodular and strictly positive, then the resulting (unique) core matching is *assortative*. That is, it matches the "best" man with the "best" woman, the second best man with the second best woman, and so forth. The converse of this result is also easily seen to be true; that is, if a unique core matching is assortative, then it can be rationalized with a strictly supermodular assignment game. Simply let α be any strictly supermodular assignment game and note that there is a unique assortative core matching, which is the matching under consideration. Generalizing this result to the case of different cardinalities of men and women, and the case of weakly supermodular and potentially negative α is an open question which is amenable to linear programming analysis.

Related is the question which asks, given an assignment game α, which sets of ordinal preferences are compatible with it? This question only makes sense if we explicitly model the underlying transferable utility model which defines α. One standard example is when M is a set of buyers of objects, and W is a set of sellers of objects. Each $m \in W$ has a valuation $u_m(w)$ of the object $w \in W$ is selling. Each $w \in W$ has a valuation v_w of the object she sells. The surplus $\alpha_{m,w}$ is then obviously $u_m(w) - v_w$. The question is then, given α, which lists of preferences $(P(m))_{m \in M}$ have utility representations u_m which generate α? Further questions might be asked as to when is it the case that $v_w(w) \geq 0$ (that is, when is it the case that each seller would rather keep her object than throw it away?) Note that in particular that a utility representation must satisfy the requirement that $u_m(m) = 0$. This question can also be addressed using linear programming techniques. Similarly, one can ask a related question about synergistic matching, in which each $m \in M$ and each $w \in W$ has preferences, and the surplus associated with utility functions is given by $\alpha_{m,w} = u_m(w) + u_w(m)$.

Also related is the question of efficient sets of matchings when preferences are not necessarily quasilinear, which again only makes sense in a model where preferences over objects sold are explicitly modeled. Such models are related to Alkan, Demange, and Gale (1991) (for example).

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